Developing Algebraic Reasoning from Quantitative Reasoning

Bruce Arnold barnold@ucsd.edu

UCSD Math Testing & Placement

Dr. Osvaldo Soto osoto@sandi.net, osoto@ucsd.edu

San Diego Unified School District/Math for America San Diego

Goals of this Presentation

To explore the development of algebraic thinking through quantitative reasoning:

- Rate problems as a tool: They are not all created equally
- How to create these problems
  - Spoiler alert: DON'T create … find and tweak
  - Again, context matters…
- How they are used with HoM, WoT, SMP's in mind
- What student thinking can look like

From Macro to Micro (or Theory and Practice)

Influential Sources:

- Usiskin's Definition of Algebra
- Pat Thompson's Theory of Quantitative Reasoning (2011)
- Harel's Necessity Principle (Harel, 2013a and 2013b)
- Fostering Algebraic Thinking (Driscoll et al)
- CCSS-M Appendix A, p. 17.

Harel (Research Perspective for CCSS)

Quantitative reasoning is a way of thinking by which one reasons with quantities and about relations among quantities. It entails the habits of creating a coherent image of the problem at hand; considering the units involved; continually attending to the meaning of quantities, in addition to how to compute them; and having multiple images of a concept and being flexible in transitioning among them.

Reason Abstractly and Quantitatively

CCSS MP2

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Creating Problems with SMP in Mind

1. "Friendly numbers" can be used to help create a coherent image of the problem.
2. Making the number "less friendly" can help focus attention on structure and meaning.
3. A context with units can be used to help attend to meaning.
4. The ability to change perspectives (or images) of a problem is essential.
5. Writing different expressions representing the same quantity can necessitate algebraic manipulation and help students attend to the meaning of the equal sign.
6. *An approach to introducing algebraic reasoning:*
    1. Noticing a pattern from repeated observations with attention to causality.
    2. Change the way you ask your question… "At any in point in time".
    3. Build scaffolds to support repeated reasoning before asking to solve…

Flooded Basement Problems (FBP's) – 35 min

Flooded Basement Problem: My basement flooded and there are 2.5 inches of water in it. Last time when it flooded there was 3/8 inch of water, and it took my pump 45 minutes to pump it out.

(1) How long will it take this time?

(2) When I started my pump, I realized that the pump has enough gasoline for one hour. According to the pump manual, the capacity of the gasoline tank is 0.5 gallon, which is sufficient for 5 hours work. What is the least amount of gasoline I need to add to my pump to ensure that it can pump all of the water out of my basement?

(3) Unfortunately, I have only 1/5 gallon of gasoline to put in my pump. My neighbor has a portable pump which holds the same amount of gasoline as my pump and pumps water out at the same rate. It has ¼ gallon of gasoline. Using all of the gasoline available, would the two pumps be able to pump 100% of the water out of my basement?

(4) If there were enough gasoline and if the two pumps were working together, how long would it take them to pump 100% of the water out of my basement?

(5) My neighbor isn't home, so I have to borrow a pump from my brother. My brother's pump pumps out water at a rate 4/5 as fast as my pump. If the two pumps were working together, how many inches of water would be pumped out of the basement at any given moment from the start of their working together?

(6) If my and my brother's pumps were working together, how long would it take them to pump all of the water out of the basement?

Flooded Basement Problems (FBP's)
Focus Problems

Flooded Basement Problem: My basement flooded and there are 2.5 inches of water in it. Last time when it flooded there was 3/8 inch of water, and it took my pump 45 minutes to pump it out.

Possible approaches/lessons to FBP – Repeated Addition Approach

Tabulate with purpose and organization, change forms, switch perspectives, persist

Number of Min's	Number of inches	Goal 2.5 inches			
45	$\frac{3}{8}$ inches				
90	$\frac{6}{8}$ inches				
135	$1\frac{1}{8}$ inches or $\frac{9}{8}$ inches				
180	$1\frac{4}{8}$ inches or $\frac{12}{8}$ inches				
225	$1\frac{7}{8}$ inches or $\frac{15}{8}$ inches	2.5 inches	$2\frac{1}{2}$ inches	$2\frac{4}{8}$ inches	$\frac{20}{8}$ inches
270	$2\frac{2}{8}$ inches or $\frac{18}{8}$ inches	2.5 inches	$2\frac{1}{2}$ inches	$2\frac{4}{8}$ inches	$\frac{20}{8}$ inches
315	$2\frac{5}{8}$ inches or $\frac{21}{8}$ inches	2.5 inches	$2\frac{1}{2}$ inches	$2\frac{4}{8}$ inches	$\frac{20}{8}$ inches
15	$\frac{1}{8}$ inches				

Possible approaches/lessons to FBP – Repeated Subtraction Approach

Repeated subtraction with decimal approximation:

$$2.5 inches - {}^{3}\!/_{8}\, inch = 2.5 - 0.375 inches = 2.125 inches$$

$$2.125 inches - {}^{3}\!/_{8}\, inch = 2.125 - 0.375 inches = 1.75 inches$$

$$1.75 inches - {}^{3}\!/_{8}\, inch = 1.75 - 0.375 inches = 1.375 inches$$

$$1.375 - {}^{3}\!/_{8}\, inch = 1.375 - 0.375 inches = 1 inch$$

$$1 - {}^{3}\!/_{8}\, inch = 1 - 0.375 inches = .625 inches$$

$$.625 - {}^{3}\!/_{8}\, inch = .625 - 0.375 inches = .25 inches$$

$$.25 - {}^{3}\!/_{8}\, inch = .25 - 0.375 inches = ??$$

– It takes longer than 6(45) minutes… How much?

Possible approaches/lessons to FBP – Proportional Reasoning

- Immediately reason that $\frac{3}{8}$ inches every 45 min's means $\frac{1}{8}$ inches every 15 min.
- Realizing that $2.5 inches = \frac{20}{8} inches$ and thinking of 1/8 as an object rather than a process.
- Note: Greg and Frances observed this as the most popular approach taken by incoming freshmen and sophomores in different settings.

Referential Versus Non-referential Symbolic Reasoning

- Compare these two approaches … both correct:

Approach 1:

Let x be the number of minutes it takes to remove 2.5 inches of water.

$$\frac{\frac{3}{8}\,inches}{45\,\text{min}}\left(x\,\text{min}\right)=\frac{20}{8}\,inches$$

$$x=\frac{20}{3}\left(45\,\text{min}\right)=300\,\text{min}$$

Approach 2:

$$\frac{\frac{3}{8}}{45}=\frac{\frac{20}{8}}{x}$$

$$x=\frac{20}{3}\left(45\right)=300$$

FBP’s #1

Some lesson so far:

- Tabulation as a tool helps communicate and organize what I think is happening in this problem
- Representations indicate mental images
- Proportional reasoning can be represented in several ways

FBP’s #2

(2) When I started my pump, I realized that the pump has enough gasoline for one hour. According to the pump manual, the capacity of the gasoline tank is 0.5 gallon, which is sufficient for 5 hours work. What is the least amount of gasoline I need to add to my pump to ensure that it can pump all of the water out of my basement?

FBP's #3

(3) Unfortunately, I have only 1/5 gallon of gasoline to put in my pump. My neighbor has a portable pump which holds the same amount of gasoline as my pump and pumps water out at the same rate. It has ¼ gallon of gasoline. Using all of the gasoline available, would the two pumps be able to pump 100% of the water out of my basement?

FBP’s #4

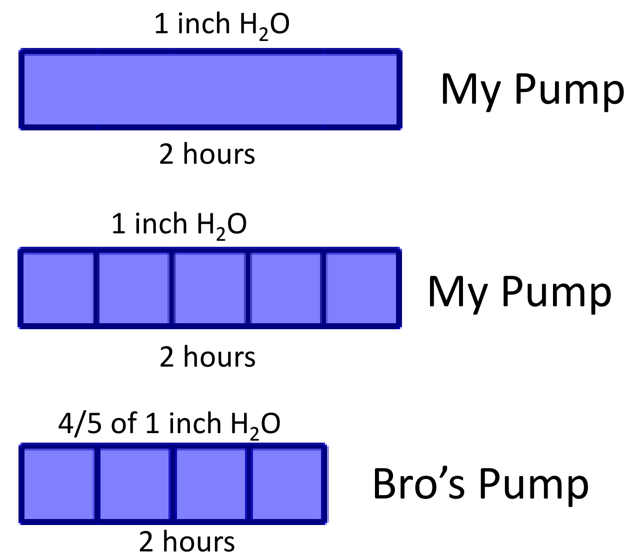
(4) If there were enough gasoline and if the two pumps were working together, how long would it take them to pump 100% of the water out of my basement?

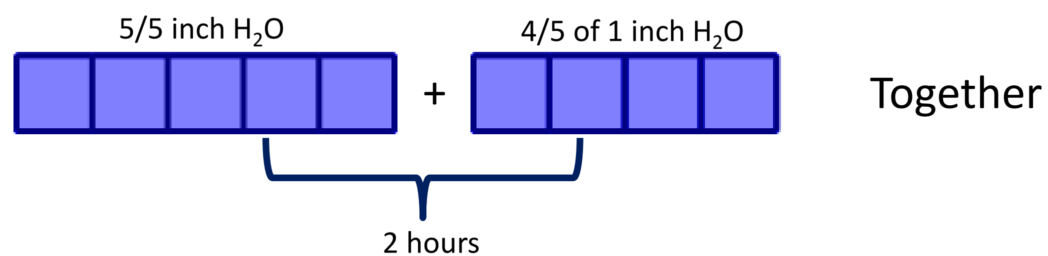
FB’s #5

(5) My neighbor isn’t home, so I have to borrow a pump from my brother. My brother’s pump pumps out water at a rate 4/5 as fast as my pump. If the two pumps were working together, how many inches of water would be pumped out of the basement at any given moment from the start of their working together?

What does "4/5 as fast as my pump" mean?

	Hold time steady and remove 4/5 as many inches H_2O…
My pump	$\frac{3/8\ inch}{45\ min} = \frac{1/8\ inch}{15\ min} = \frac{1/2\ inch}{1\ hour} = \frac{1\ inch}{2\ hours}$
My brother's pump	$\frac{4/5\ inch}{2\ hours}$
Together	$\frac{1\ inch}{2\ hours} + \frac{4/5\ inch}{2\ hours} = \frac{9/5\ inch}{2\ hours} = ?$





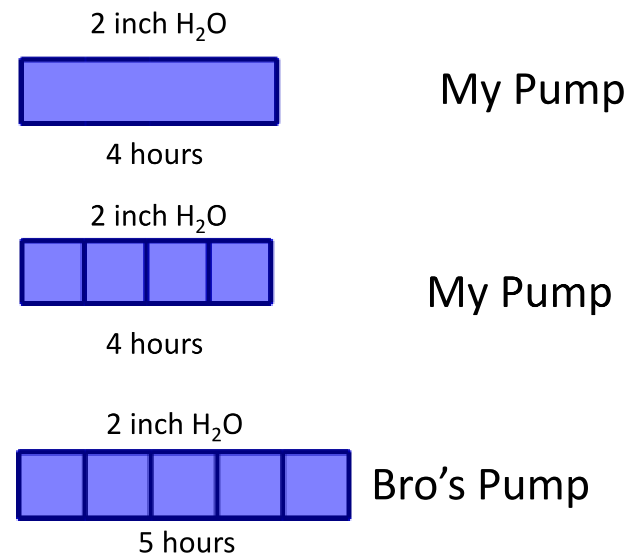
At any point in time: Targeted versus observed solutions

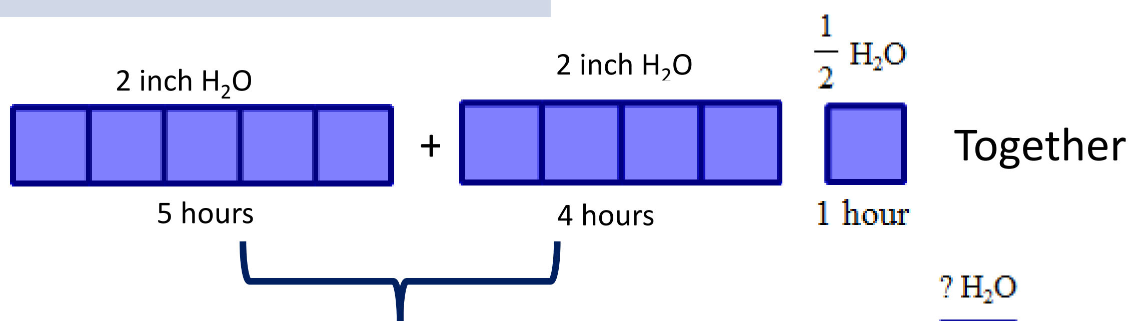
Targeted Answer: f(t) = (9/10) t where t is in hours.

Observed Answers: Provide a table

What does “4/5 as fast as my pump” mean?

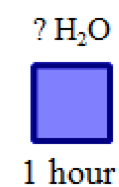
	Hold inches H_2O steady and use 5/4 as much time…
My pump	$\frac{{}^3/_8\ inch}{45\ min} = \frac{{}^1/_8 inch}{15\ min} = \frac{{}^1/_2 inch}{1\ hour} = \frac{2\ inch}{4\ hours}$
My brother’s pump	$\frac{2\ inch}{5\ hours}$
Together	$\frac{2\ inch}{5\ hours} + \frac{2\frac{1}{2}\ inch}{5\ hours} = \frac{4\frac{1}{2}\ inch}{5\ hours}$ $\frac{2\ inch}{5\ hours} + \frac{2\ inch}{4\ hours} = ?$





New Problem →

How does this work when times are unequal?



To students an answer is …

- The domain is whole number of hours… Therefore the list is the answer.
- Consistent with image of function as a set of points - Several ways to handle this pedagogically.
- In problem 6 there will be a need to refine this table b/c $^2/_{10}\ inch$ too much water has been pumped out at 3 hours.
- How do we necessitate the targeted solution? This is one way to use problem #6.

Time (hrs)	Mine (in.)	Bro's (in.)	Together(in.)
1	$\frac{1}{2}\ inch = \frac{5}{10}\ inch$	$\frac{2}{5}\ inch = \frac{4}{10}\ inch$	$\frac{9}{10}\ inch$
2	$\frac{10}{10}\ inch$	$\frac{8}{10}\ inch$	$\frac{18}{10}\ inch$
3	$\frac{15}{10}\ inch$	$\frac{12}{10}\ inch$	$\frac{27}{10}\ inch$
4	$\frac{20}{10}\ inch$	$\frac{16}{10}\ inch$	$\frac{36}{10}\ inch$
5	$\frac{25}{10}\ inch$	$\frac{20}{10}\ inch$	$\frac{45}{10}\ inch$

FBP’s #6

(6) If my and my brother’s pumps were working together, how long would it take them to pump all of the water out of the basement?

Problem 6

Time (hrs)	Mine (in.)	Bro's (in.)	Together(in.)
1	$\frac{1}{2}$ $inch$= $\frac{5}{10}$ $inch$	$\frac{2}{5}$ $inch$ $=\frac{4}{10}$ $inch$	$\frac{9}{10}$ $inch$
2	$\frac{10}{10}$ $inch$	$\frac{8}{10}$ $inch$	$\frac{18}{10}$ $inch$
3	$\frac{15}{10}$ $inch$	$\frac{12}{10}$ $inch$	$\frac{27}{10}$ $inch$
4	$\frac{20}{10}$ $inch$	$\frac{16}{10}$ $inch$	$\frac{36}{10}$ $inch$
5	$\frac{25}{10}$ $inch$	$\frac{20}{10}$ $inch$	$\frac{45}{10}$ $inch$

- 3 hours → $\frac{2}{10}$ inch <u>too much</u> was removed
- Every hour together we remove $\frac{9}{10}$ inch
- In 6 min we remove $\frac{1}{10}$ inch → 12 min removes $\frac{2}{10}$ inch
- So, it takes us 2 hours and 48 min together

Quantities → Structures → Algebra

How can these problems help lay the foundation for algebraic reasoning?

- Reinforcing a need to attend to meaning:
  - Precise use of units of measure in a context
  - Attention to relationships between quantities
  - Massage wording: Phenomenon BEFORE label … “At any point in time”.
- Importance of seeing 4/5 as a part-to-whole relationship or as a ratio of 4 to 5 (4:5).
- Structure: Why do you keep the denominator when adding fractions?
- Functional way of thinking
  - Rate in terms of inches
  - Rate in terms of time
- Pedagogical notes:
  - Habits of mind take time to form
  - Teacher’s role: Attention to domain
  - Teacher’s role: Point out mathematical values

Special Thanks to…

- Guershon Harel (PI/PD’er/Author)
- Greg Guayante (MfASD MTF)

References

- Driscoll, Mark. *Fostering Algebraic Thinking: A Guide for Teachers, Grades 6-10*. Heinemann, 361 Hanover Street, Portsmouth, NH 03801-3912, 1999.
- Harel, G. (2013a). Intellectual Need. In *Vital Direction for Mathematics Education Research*, Leatham, K. Ed., Springer.
- Harel, G. (2013). The Kaputian program and its relation to DNR-based instruction: A common commitment to the developmnent of mathematics with meaning, In *The SimCalc Vision and Contribution,* (Fried, M., & Dreyfus, T., Eds.), Springer, 438-448.
- Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* WISDOMe Monographs (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming Press.
- Usiskin, Zalman. “Conceptions of School Algebra and Uses of Variables.” In Algebraic Thinking, Grades K–12: Readings from NCTM’s School-Based Journals and Other Publications, edited by Barbara Moses, pp. 7–13. Reston, Va.: National Council of Teachers of Mathematics, 1999.

- For more information on Harel’s DNR Theoretical, visit http://www.math.ucsd.edu/~harel
- For more information on Thompson’s Theory of Quantitative Reasoning, visit http://pat-thompson.net/
- Flooded Basement (and other holistic problems) http://www.math.ucsd.edu/~harel/projects/Downloadable/Holistic%20Problems.pdf

Usiskin: Conceptions of Algebra

1. Generalized arithmetic
2. A study of procedures for solving certain kinds of problems
3. The study of relationships among quantities
4. Algebra as the study of structures

Coordinating Information

Here are some things I know, but they might be scattered thoughts…

I want to pump out 2.5 inches

The pump has enough gas for 1 hour

5 hrs of work takes .5 gallons

Want to pump for 5 hours

What is the least amount of gas I have to add?

Possible approach

- .5 gallons → 5 hrs. of work
- .1 gallons → 1 hr. of work

It will take 5 hrs. to pump 2.5 inches.

The pump has .1 gallons (1 hour of work)

So the pump needs .4 gallons more.

Possible Approach

Immediate focus on time/proportional reasoning

- I can run the pump for one hr. and I need to 5 hrs. So, I need 4 hrs. more.
- 4 hrs. is 4/5 of 5 hrs → 4/5 of .5 gal
- 2/5 gal needed

- It takes 5 hrs to pump out 2.5 in.
- 1 hr/5hrs → I can run the pump for 1/5 of the time I need to.

Gen's Sequence/Version of Harel's (2013) Motion Problems

1) From two towns, A and B, two cars left at the same time toward each other – one from Town A and the other from Town B. The speed of the first car is 70 mph, and the speed of the second car is 80 mph. If the distance between the two towns is 300 miles, how long does it take for the two cars to meet?

2) A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 12 mph, and the motor cycler at 40 mph. In how many hours will the distance between them be more than 105 miles?

3) A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 15 mph, and the motor cycler at 55 mph. In how many hours will the distance between them be more than 170 miles?

Building Algebraic Thinking through Quantitative Reasoning: Applying Harel's Necessity Principle

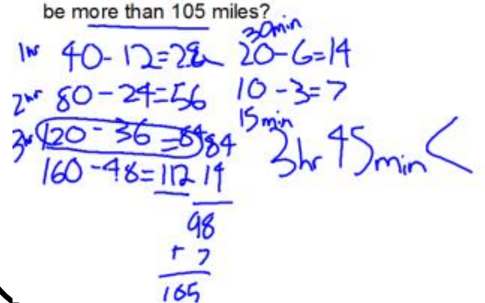
Sequence for Distance Problems:

1) From two towns, A and B, two cars left at the same time toward each other – one from Town A and the other from Town B. The speed of the first car is 70 mph, and the speed of the second car is 80 mph. How long does it take for the two cars to meet, if the distance between the two towns is 300 miles? 150 miles? 525 miles? 273 miles?

2) A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 12 mph, and the motor cycler at 40 mph. In how many hours will the distance between them be more than 56 miles? 140 miles? 42 miles? 105 miles?

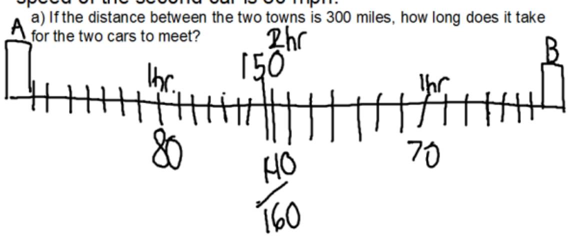
2) A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 12 mph, and the motor cycler at 40 mph.

c) In how many hours will the distance between them be more than 105 miles?



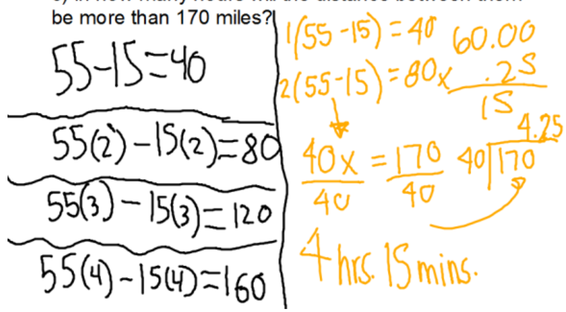
From two towns, A and B, two cars left at the same time toward each other – one from Town A and the other from Town B. The speed of the first car is 70 mph, and the speed of the second car is 80 mph.

a) If the distance between the two towns is 300 miles, how long does it take for the two cars to meet?



A biker and a motor cycler left from the same location at the same time, and in the same direction; the biker at the speed of 15 mph, and the motor cycler at 55 mph.

c) In how many hours will the distance between them be more than 170 miles?



- "Friendly" solutions first.
- Opportunities to use knowledge of motion.
- Changing to less "friendly" solutions allows repeated reasoning about the meaning of quantities until a pattern was noticed
- An equation based on the quantity that is varying was written.

Difference between Representation and Problem Solving Approaches

- Form versus substance
  - Representations are useful, but not the point
- Student work shows:
  - Double number line
  - Tables and pseudo-tables
  - Equations
- Closed versus open computations
  - Open form emphasizes structure