# Algebraic Thinking Through the Grades

## Mathematics: Discipline and Curriculum

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May 30, 2013

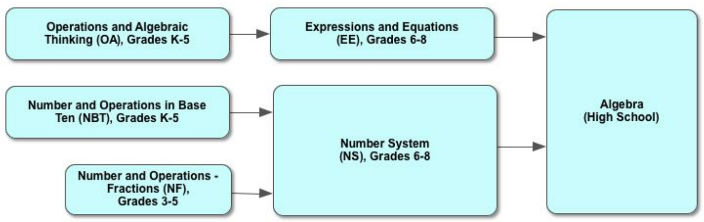
## Overview of today's session

- Consider mathematics as a discipline
- Characterize curricula aligned with a disciplinary view
- Analyze examples from a sample Common Core State Standards *Progression*
- Move forward

## Mathematics as a discipline

- Mathematical ideas are connected in multiple ways
- There are natural flows of ideas in mathematics
- Mathematics is about relationships and structure
- One way to make sense of mathematics is to understand the relationships and structure

## Example: Major Flows Leading to Algebra in CCSS-M




For example multiplication: moving from $3^{rd}$ grade to $6^{th}$ grade (and further), discrete to continuous

Equal groups → "times as much" → scale factor → unit rates

Jason Zimba, Examples of Structure in the *Common Core State Standards'* Standards for Mathematical Content

## Math curriculum should reflect the nature of mathematics

"There are many ways to organize curricula. The challenge, now rarely met, is to avoid those that distort mathematics and turn off students." Steen, 2007

## What would characterize such a math curriculum?

- Coherence
  ...articulated over time as a sequence of topics and performances that are logical and reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives. (Schmidt & Houang, 2002)
- Structure
  a set of content standards must evolve from particulars (e.g., the meaning and operations of whole numbers, ...) to deeper structures inherent in the discipline. (Schmidt & Houang 2002)
- Focus
  Fewer standards, deeper (not "a mile wide and inch deep")

http://www.corestandards.org/Math

## What is Algebra?

One attempt to bridge K-12 and university algebra:

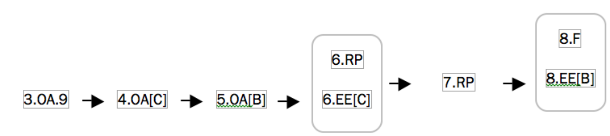
> *Algebra starts as the art of manipulating sums, products, and powers of numbers. The rules for these manipulations hold for all numbers, so the manipulations may be carried out with letters standing for the numbers. It then appears that the same rules hold for various different sorts of numbers and that the rules even apply to things. which are not numbers at all. An* algebraic system, *as we will study it, is thus* a set of elements of any sort on which functions such as addition and multiplication operate, provided only that these operations satisfy certain basic rules.

From *Algebra* by Mac Lane and Birkhoff (1967)

Arithmetic $\rightarrow$ Variables $\rightarrow$ System with operations (functions)

## Sample CCSS *Progression*: *Patterns* $\rightarrow$ *Sequences* $\rightarrow$ *Functions*

Functional Thinking Stream




Functional Thinking. Capital letters in square brackets refer to clusters. For example, 6.EE[C] refers to the third cluster in 6.EE.

http://ime.math.arizona.edu/progressions/

## STANDARDS FROM THE PROGRESSION
## *PATTERNS --> SEQUENCES --> FUNCTIONS*

3.OA

9. Identify arithmetic patterns (including patterns in the addition table or multiplication table), and explain them using properties of operations. *For example, observe that 4 times a number is always even, and explain why 4 times a number can be decomposed into two equal addends.*

4.OA

5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself. *For example, given the rule "Add 3" and the starting number 1, generate terms in the resulting sequence and observe that the terms appear to alternate between odd and even numbers. Explain informally why the numbers will continue to alternate in this way.*

5.OA

3. Generate two numerical patterns using two given rules. Identify apparent relationships between corresponding terms. Form ordered pairs consisting of corresponding terms from the two patterns, and graph the ordered pairs on a coordinate plane. *For example, given the rule "Add 3" and the starting number 0, and given the rule "Add 6" and the starting number 0, generate terms in the resulting sequences, and observe that the terms in one sequence are twice the corresponding terms in the other sequence. Explain informally why this is so.*

6.EE

9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.

7.RP

2. Recognize and represent proportional relationships between quantities.
   a. Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.
   b. Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.
   c. Represent proportional relationships by equations. For example, if total cost $t$ is proportional to the number $n$ of items purchased at a constant price $p$, the relationship between the total cost and the number of items can be expressed as $t = pn$.
   d. Explain what a point $(x, y)$ on the graph of a proportional relationship means in terms of the situation, with special attention to the points $(0, 0)$ and $(1, r)$ where $r$ is the unit rate.

8.F

Define, evaluate, and compare functions.

1. Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.

http://ime.math.arizona.edu/progressions/

2. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.
3. Interpret the equation y = mx + b as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. For example, the function A = s2 giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line

Use functions to model relationships between quantities.

4. Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
5. Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

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**Analysis of a Problem**

For the following problem

1. Solve the problem
2. Consider the sequence of standards in the handout (from both progressions).
   a. What is the "target" standard of this problem?
   b. What sequence of standards from previous grads supports students in solving this problem?
   c. What standard(s) for mathematical practice are most apparent either in support of solving the problem, or as standard(s) to be developed through working on the problem?
3. How is this problem different from the typical problems given today to high school students working on exponential functions?

**Lake Algae**

On June 1, a fast growing species of algae is accidentally introduced into a lake in a city park. It starts to grow and cover the surface of the lake in such a way the the area covered by the lake will doubles every day. If it continues to grow unabated, the lake will be totally covered and the fish in the lake will suffocate. At the rate it is growing, this will happen on June 30.

a. When will the lake be covered half way?
b. On June 26, a pedestrian who walks by the lake every day warns that the lake will be completely covered soon. Her friend just laughs. Why might her friend be skeptical of the warning?
c. On June 29, a clean-up crew arrives at the lake and removes almost all of the algae. When they are done, only 1% of the surface is covered with algae. How well does this solve the problem of the algae in the lake?
d. Write an equation that represents the percentage of the surface area of the lake that is covered in algae as a function of time (in days) that passes since the algae was introduced into the lake if the cleanup crew does not come on June 29.

http://www.illustrativemathematics.org/illustrations/533